Harvard College

# Math 21b: Linear Algebra and Differential Equations

## Formula and Theorem Review

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# Contents

# 1 Linear Equations

## 1.1 Standard Representation of a Vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

## 1.2 Reduced Row Echelon Form

- If a column contains a leading 1, then all ther other entires in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

## 1.3 Elementary Row Operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.

## 1.4 Rank of a Matrix

The rank of a matrix $A$ is the number of leading 1's in $rref(A)$.

## 1.5 Dot Product of Vectors

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$$

## 1.6 The Product $A\vec{x}$

$$A\vec{x} = \begin{bmatrix} \cdots & \vec{w}_1 & \cdots \\ & \vdots & \\ \cdots & \vec{w}_n & \cdots \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

## 1.7 Algebraic Rules for $A\vec{x}$

- $A(\vec{x}\vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$

# 2 Linear Transformations

## 2.1 Linear Transformations

A function $T$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if there exists a matrix $A$ such that

$$T(\vec{x}) = A\vec{x}$$

A transformation is called linear if and only if

- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
- $T(k\vec{v}) = kT(\vec{v})$

## 2.2 Scaling Matrix

A scaling matrix by $k$ has the form:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

## 2.3 Orthogonal Projection onto a Line

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}$$

where $\vec{w}$ is a vector parallel to the line $L$.

## 2.4 Reflection Matrix

A reflection matrix has the form:

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

## 2.5 Rotation Matrix

A rotation matrix has the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

## 2.6 Shear Matrix

A horizontal shear matrix has the form:

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and a vertical shear matrix has the form:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

## 2.7 Matrix Multiplication

The product of matrices $BA$ is defined as the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$.

- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(kA)B = A(kB) = k(AB)$

## 2.8 Invertibility

An $n \times n$ matrix $A$ is invertible if and only if:

1. $rref(A) = I_n$
2. $rank(A) = n$

or, more simply, if and only if $det(A) \neq 0$.

## 2.9 Finding the Inverse

1. Form the $n \times (2n)$ matrix $[A|I_n]$
2. Compute $rref[A|I_n]$
3. If $rref[A|I_n]$ is of the form $[I_n|B]$ then $A^{-1} = B$
4. If $rref[A|I_n]$ is of another form, then $A$ is not invertible

## 2.10 Properties of Invertible Matrices

If an $n \times n$ matrix $A$ is invertible:

- The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for all $\vec{b}$ in $\mathbb{R}^n$
- $rref(A) = I_n$
- $rank(A) = n$
- $im(A) = \mathbb{R}^n$
- $ker(A) = \vec{0}$
- The column vectors of $A$ are linearly independent and form a basis of $\mathbb{R}^n$

# 3 Subspaces of $\mathbb{R}^n$ and Their Dimensions

## 3.1 Image of a Function

$$image(f) = \{f(x) : x \text{ in } X\} = \{b \text{ in } Y : b = f(x), \text{ for some } x \text{ in } X\}$$

For a linear transformation $T$:

- The zero vector is in the image of $T$
- If $\vec{v}_1$ and $\vec{v}_2$ are in the image of $T$, then so is $\vec{v}_1 + \vec{v}_2$
- If $\vec{v}$ is in the image of $T$, then so is $k\vec{v}$

## 3.2 Span

$$span(\vec{v}_1, \cdots, \vec{v}_n) = \{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m : c_1 \cdots c_n \text{ in } \mathbb{R}\}$$

## 3.3 Kernel

The kernel of a linear transformation $T$ is the solution set of the linear system:

$$A\vec{x} = \vec{0}$$

- The zero vector is in the kernel of $T$
- If $\vec{v}_1$ and $\vec{v}_2$ are in the kerne; of $T$, then so is $\vec{v}_1 + \vec{v}_2$
- If $\vec{v}$ is in the kernel of $T$, then so is $k\vec{v}$

## 3.4 Subspaces of $\mathbb{R}^n$

A subset $W$ of the vector space $\mathbb{R}^n$ is a linear subspace if and only if:

1. $W$ contains the zero vector
2. $W$ is closed under addition
3. $W$ is closed under scalar multiplication

The image and kernel of a linear transformation are linear subspaces.

## 3.5 Linear Independence

1. If a vector $\vec{v}_i$ in $\vec{v}_1, \cdots, \vec{v}_n$ can be expressed as a linear combination of the vectors $\vec{v}_1, \cdots, \vec{v}_{i-1}$, then $\vec{v}_i$ is redundant.
2. The vectors $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent if no vector is redundant.
3. The vectors $\vec{v}_1, \cdots, \vec{v}_n$ form a basis of a subspace $V$ if they span $V$ and are linearly independent

## 3.6 Dimension

The number of vectors in a subspace $V$ is the dimension of $V$.

## 3.7 Rank-Nullity Theorem

For an $n \times m$ matrix $A$:

$$dim(kerA) + dim(imA) = m$$

## 3.8 Coordinates

For a basis $\mathfrak{B} = (\vec{v}_1, \cdots, \vec{v}_n)$ of a subspace $V$, any vector $\vec{x}$ can be written as:

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

$c_1, \cdots, c_n$ are called the $\mathfrak{B}$-coordinates of $\vec{x}$, with

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}} \quad \text{and} \quad [\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x}$$

where $S = [v_1 \cdots v_n]$

## 3.9 Linearity of Coordinates

- $[\vec{x} + \vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}}$
- $[k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}}$

## 3.10 Matrix of a Linear Transformation

The matrix $B$ that transforms $[\vec{x}]_{\mathfrak{B}}$ into $[T(\vec{x})]_{\mathfrak{B}}$ is called the $\mathfrak{B}$-matrix of $T$:

$$[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}, \quad \text{where } B = [\,[T(\vec{v}_1)]_{\mathfrak{B}} \cdots [T(\vec{v}_n)]_{\mathfrak{B}}\,]$$

## 3.11 Similar Matrices

Two matrices $A$ and $B$ are similar if and only if:

$$AS = SB \quad \text{or} \quad B = S^{-1}AS$$

# 4 Linear Spaces

## 4.1 Linear Spaces

A linear space $V$ is a set with rules for addition and scalar multiplication that satisfies the following properties:

1. $(f + g) + h = f + (g + h)$
2. $f + g = g + f$
3. There exists a neutral element $n$ in $V$ such that $f + n = f$
4. For each $f$ in $V$ there exists a $g$ in $V$ such that $f + g = 0$
5. $k(f + g) = kf + kg$
6. $(c + k)f = cf + kf$
7. $c(kf) = (ck)f$
8. $1f = f$

## 4.2 Finding a Basis of a Linear Space $V$

1. Find a typical element $w$ of $V$ in terms of arbitrary constants.
2. Express $w$ as a linear combination of elements in $V$.
3. If these elements are linearly independent, they will for a basis of $V$.

## 4.3 Isomorphisms

- A linear transformation $T$ is an isomorphism of $T$ is invertible.
- A linear transformation $T$ from $V$ to $W$ is an isomorphism if any only if $ker(T) = 0$ and $im(T) = W$.
- Coordinate changes are isomorphisms.
- If $V$ is isomorphic to $W$, then $dim(V) = dim(W)$.

# 5 Orthogonality and Least Squares

## 5.1 Orthongality and Length

- Two vectors $\vec{c}$ and $\vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.
- The length of a vector $\vec{v}$ is $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$.
- A vector $\vec{u}$ is a unit vector if its length is 1.

## 5.2 Orthonormal Vectors

The vectors $\vec{u}_1, \cdots, u_n$ are orthonormal if they are unit vectors orthogonal to each other.

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

## 5.3 Orthogonal Projection onto a Subspace

If $V$ is a subspace with an orthonormal basis $\vec{u}_1, \cdots, \vec{u}_n$:

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$$

## 5.4 Orthogonal Complement

The orthogonal complement $V^\perp$ of a subspace $V$ is the set of vectors $\vec{x}$ orthogonal to all vectors in $V$:

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \text{ for all } \vec{v} \text{ in } V\}$$

- $V^\perp$ is a subspace of $V$
- $V \cap V^\perp = \vec{0}$
- $dim(V) + dim(V^\perp) = n$
- $(V^\perp)^\perp = V$

## 5.5 Pythongorean Theorem

$$||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$$

## 5.6 Cauchy-Schwarz Inequality

$$||\vec{x} \cdot \vec{y}|| \leq ||\vec{x}||||\vec{y}||$$

## 5.7 Angle Between Two Vectors

$$\theta = \arccos \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||||\vec{y}||}$$

## 5.8 Gram-Schmidt Process

For a basis $\vec{v}_1, \cdots, \vec{v}_n$ of a subspace $V$:

$$\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1, \cdots, \vec{u}_n = \frac{1}{||\vec{v}_n^{\perp}||}$$

where

$$\vec{v}_i^{\perp} = \vec{v}_i - (\vec{u}_1 \cdot v_i)\vec{u}_1 - \cdots - (\vec{u}_{i-1} \cdot v_i)\vec{v}_{i-1}$$

## 5.9 QR Decomposition

For an $n \times m$ matrix $M$, $M = QR$, where $Q$ is an $n \times m$ matrix whose columns $\vec{u}_1, \cdots, \vec{u}_n$ are orthonormal and $R$ has entries satisfying:

$$r_1 1 = ||\vec{v}_1||, \quad r_{jj} = ||\vec{v}_j^{\perp}||, \quad r_{ij} = \vec{u}_i \cdot \vec{v}_j \text{ for } i < j$$

## 5.10 Orthongonal Transformation

A linear transformation $T$ is considered orthogonal if it preserves the length of vectors, such that

$$||T(\vec{x})|| = ||\vec{x}||$$

- $T$ is orthogonal if the vectors $T(\vec{e}_1), \cdots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$.
- The matrix $A$ is orthogonal if $A^T A = I_n$.
- The matrix $A$ is orthogonal if $A^{-1} = A^T$.
- A matrix $A$ is orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$.
- The product $AB$ of two orthogonal matrices $A$ and $B$ is orthogonal.
- The inverse $A^{-1}$ of an orthogonal matrix $A$ is orthogonal.

## 5.11 Transpose

The transpose $A^T$ of an $m \times n$ matrix $A$ is the $n \times m$ matrix whose $ij$th entry is the $ji$th entry of $A$.

- $(AB)^T = B^T A^T$
- $(A^T)^{-1} = (A^{-1})^T$
- $rank(A) = rank(A^T)$.

## 5.12 Symmetric and Skew Symmetric Matrices

- An $n \times n$ matrix $A$ is symmetric if $A^T = A$.
- An $n \times n$ matrix $A$ is skew-symmetric if $A^T = -A$.

## 5.13 Matrix of an Orthogonal Projection

The orthongonal projection onto a subspace $V$ with an orthonormal basis $\vec{u}_1, \cdots, \vec{u}_n$ is

$$QQ^T, \quad \text{where } Q = [\vec{u}_1 \cdots \vec{u}_n]$$

or equivalently,

$$A(A^TA)^{-1}A^T \quad \text{where } A = [\vec{v}_1 \cdots \vec{v}_n]$$

## 5.14 Least-Squares Solution

The unique least-squares solution of a linear system $A\vec{x} = \vec{b}$ where $ker(A) = \vec{0}$ is

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

## 5.15 Inner Product Spaces

The inner product of a linear space $V$, denoted $\langle f, g\rangle$, has the following properties:

- $\langle f, g\rangle = \langle g, f\rangle$
- $\langle f + h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$
- $\langle cf, g\rangle = c\langle f, g\rangle$
- $\langle f, f\rangle > 0$

## 5.16 Norm and Orthongonality

The norm of an element $f$ of an inner product space is

$$||f|| = \sqrt{\langle f, f\rangle}$$

Two elements $f$ and $g$ of an inner product space are orthongonal if

$$\langle f, g\rangle = 0$$

## 5.17 Trace of a Matrix

The trace $tr(A)$ of a matrix $A$ is the sum of its diagonal entries.

### 5.18 Orthogonal Projection of an Inner Product Space

If $g_1, \cdots, g_n$ is an orthonormal basis of a subspace $W$ of an inner product space $V$:

$$\text{proj}_W f = \langle g_1, f \rangle g_1 + \cdots + \langle g_n, f \rangle g_n$$

### 5.19 Fourier Analysis

$$f_n(t) = a_0 \frac{1}{\sqrt{2}} + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt)$$

where

$$a_0 = \langle f, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} f(t)\ dt$$

$$b_k = \langle f, \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\ dt$$

$$c_k = \langle f, \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\ dt$$

## 6 Determinants

### 6.1 Sarrus's Rule

For an $n \times n$ matrix $A$, write the first $n-1$ columns to the right of $A$, then multiply along the diagonal to get $2n$ products. Subtract the first $n$ products, then add the second $n$ products to get the determinant.

### 6.2 Patterns

- A pattern of an $n \times n$ matrix $A$ is a way to choose $n$ entries of $A$ such that each entry is in a unique row and column.
- The product of a pattern is designated $P$.
- Two entries in a pattern are an inversion if one is located above and to the right of the other.
- The signature of a pattern is defined as $sgn\ P = (-1)$(inversions in P)

$$det(A) = \sum (sgn\ P)(prod\ P)$$

### 6.3 Determinants and Gauss-Jordan Elimination

If $B$ is an $n \times n$ matrix obtained from applying an elementary row operation on an $n \times n$ matrix $A$:

- If B is obtained by row division: $det(B) = \frac{1}{k} det(A)$
- If B is obtained by row swap: $det(B) = -det(A)$
- If B is obtained by row addition: $det(B) = det(A)$

### 6.4 Laplace Expansion

Expansion down the $j$th column:

$$det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$$

Expansion along the $i$th row:

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$$

### 6.5 Properties of the Determinant

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- If $A$ and $B$ are similar, then $det(A) = det(B)$
- $det(A^{-1}) = \frac{1}{det(A)} = det(A)^{-1}$

## 7 Eigenvalues and Eigenvectors

### 7.1 Eigenvalues

$\lambda$ is an eigenvalues of an $n \times n$ matrix $A$ if and only if

$$det(A - \lambda I_n) = 0$$

### 7.2 Characteristic Polynomial

$$det(A - \lambda I_n) = (-1)^n \lambda^n + (-1)^{n-1} tr(A) \lambda^{n-1} + \cdots + det(A)$$

## 7.3 Algebraic Multiplicity

An eigenvalue $\lambda$ has algebraic multiplicty $k$ if it is a root of multiplicity $k$ of the characteristic polynomial.

## 7.4 Eigenvalues, Determinant, and Trace

For an $n \times n$ matrix $A$

$$det(A) = \lambda_1 \cdots \lambda_n = \prod_{k=1}^{n} \lambda_k$$

$$tr(A) = \lambda_1 + \cdots + \lambda_n = \sum_{k=1}^{n} \lambda_k$$

## 7.5 Eigenspace

$$E_\lambda = ker(A - \lambda I_n) = \{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

## 7.6 Eigenbasis

An eigenbasis for an $n \times n$ matrix $A$ conists of the eigenvectors of $A$ and forms a basis for $\mathbb{R}^n$.

## 7.7 Geometric Multiplicty

The dimension of the eignspace $E_\lambda$ is the geometric multiplicty of the eigenvalue $\lambda$.

## 7.8 Diagonalization

1. Find the eigenvalues and corresponding eigenvectors of the matrix $A$.
2. Let $S$ be the eigenbasis for $A$ and $D$ be a matrix with the eigenvalues of $A$ along the diagonal.
3. $D = S^{-1}AS$ and $A = SDS^{-1}$

## 7.9 Powers of a Diagonalizable Matrix

If $A = SDS^{-1}$, then

$$A^t = SD^tS^{-1}$$

### 7.10 Stable Equilibrium

$\vec{0}$ is an asympototically stable equilibrium for the system $\vec{x}(t+1) = A\vec{x}(t)$ if

$$\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty} A^t = \vec{0}$$

# 9 Linear Differential Equations

### 9.1 Exponential Growth and Decay

$$\frac{dx}{dt} = kx, \quad x(t) = e^{kt}x_0$$

### 9.2 Linear Dynamical Systems

- Discrete model: $\vec{x}(t+1) = B\vec{x}(t)$
- Continuous model: $\frac{d\vec{x}}{dt} = A\vec{x}$

### 9.3 Continuous Dynamical Systems with Real Eigenvalues

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

where $\vec{v}_1, \cdots \vec{v}_n$ forms a real eigenbasis of $A$ with eignvalues $\lambda_1, \cdots, \lambda_n$

### 9.4 Continuous Dynamical Systems with Complex Eigenvalues

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(t) = e^{pt}S\begin{bmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{bmatrix}S^{-1}\vec{x}_0$$

where $p \pm q$ are eigenvalues with eigenvectors $\vec{v} \pm \vec{w}$ and $S = [\vec{w}\vec{v}]$.

### 9.5 Strategy for Solving Linear Differential Equations

To solve an nth order linear differential equation with the form $T(f) = g$:

1. Find a basis $f_1, \cdots, f_n$ of $ker(T)$.
2. Find a particular solution $f_p$.
3. The solutions $f$ are in the form $f = c_1 f_1 + \cdots + c_n f_n + f_p$.

## 9.6 Eigenfunctions

A smooth function $F$ is an eigenfunction of $T$ if

$$T(f) = \lambda f$$

## 9.7 Characteristic Polynomial of a Linear Differential Operator

For $T(f) = f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f + a_0 f$, the characteristic polynomial is defined as:

$$p_T(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

If $e^{\lambda t}$ is an eigenfunction of $T$ with eigenvalue $p_T(\lambda)$:

$$T(e^{\lambda t}) = p_T(\lambda)e^{\lambda t}$$

## 9.8 Kernel of a Linear Differential Operator

If $T$ is a linear differential operator with characteristic polynomial $p_T(\lambda)$ with roots $\lambda_1, \cdots, \lambda_n$, then the kernel of $T$ is formed by

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$$

## 9.9 Characteristic Polynomial with Complex Solution

If the zeros of $p_T(\lambda)$ are $p \pm q$, then the solutions to its differential equation are of the form

$$f(t) = e^{pt}(c_1 \cos(qt) + c_2 \sin(qt))$$

## 9.10 First-Order Linear Differential Equations

A differential equation of the form $f'(t) - af(t) = g(t)$ has a solution of the form

$$f(t) = e^{at} \int e^{-at} g(t)\ dt$$

## 9.11 Strategy for Solving Linear Differential Equations

To solve the $n$th order linear differential equation $T(f) = g$:

1. Find $n$ linearly independent solutions of $T(f) = 0$.
   - Write the characteristic polynomial $p_T(\lambda)$ of $T$ by replacing $f^{(k)}$ with $\lambda^k$
   - Find the solutions $\lambda_1, \cdots, \lambda_n$ of the equation $p_T(\lambda) = 0$.
   - If $\lambda$ is a solution of $p_T(\lambda) = 0$, then $e^{\lambda t}$ is a solution of $T(f) = 0$.

- If $\lambda$ is a solution of $p_t(\lambda) = 0$ with multiplicity $m$, then $e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \cdots, t^{m-1}e^{\lambda t}$ are the solutions of $T(f) = 0$.
- If $p \pm q$ are complex solutions of $p_T(\lambda) = 0$, then $e^{pt}\cos(qt)$ and $e^{pt}\sin(qt)$ are real solutions of $T(f) = 0$.

2. If $T(f)$ is inhomogenous, find one particular solution $f_p$ of the equation $T(f) = g$.
   - If $g$ is of the form $g(t) = A\cos(\omega t) + B\sin(\omega t)$, $g(t) = A\cos(\omega t)$, or $g(t) = A\sin(\omega t)$, look for a particular solution of the form $f_p(t) = P\cos(\omega t) + Q\sin(\omega t)$.
   - If $g$ is of the form $g(t) = a_0 + a_1 t + \cdots a_n t^n$, look for a particular solution of the form $f_p(t) = c_0 + c_1 t + \cdots c_n t^n$
   - If $g$ is constant, look for a particular solution of the form $f_p(t) = c$.
   - If $g$ is of the form $g(t) = f'(t) - af(t)$, use the formula $f(t) = e^{at}\int e^{-at}g(t)\, dt$.

3. The solutions of $T(f) = g$ are of the form

$$f(t) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) + f_p(t)$$

where $f_1, \cdots, f_n$ are the solutions from Step 1 and $f_p$ is the solution from Step 2.

## 9.12 Linearization of a Nonlinear System

If $(a, b)$ is an equilibrium point of the system $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, such that $f(a, b) = 0$ and $g(a, b) = 0$, then the system is approximated near $(a, b)$ by the Jacobian matrix:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

## 9.13 The Heat Equation

$$f_t(x, t) = \mu f_{xx}(x, t)$$

has solutions of the form

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t} \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x,0)\sin(nx)\, dx$$

## 9.14 The Wave Equation

$$f_{tt}(x, t) = c^2 f_{xx}(x, t)$$

has solutions of the form

$$f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x,0)\sin(nx)\, dx$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f_t(x,0)\sin(nx)\, dx$.