

Harvard College

Math 21a: Multivariable Calculus
FORMULA AND THEOREM REVIEW

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9 Vectors and the Geometry of Space

9.1 Distance Formula in 3 Dimensions

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

9.2 Equation of a Sphere

The equation of a sphere with center (h, k, l) and radius r is given by:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

9.3 Properties of Vectors

If \vec{a} , \vec{b} , and \vec{c} are vectors and c and d are scalars:

$$\begin{array}{ll} \vec{a} + \vec{b} = \vec{b} + \vec{a} & \vec{a} + 0 = \vec{a} \\ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} & \vec{a} + -\vec{a} = 0 \\ c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b} & (c + d)\vec{a} = c\vec{a} + d\vec{a} \\ (cd)\vec{a} = c(d\vec{a}) & \end{array}$$

9.4 Unit Vector

A unit vector is a vector whose length is 1. The unit vector \vec{u} in the same direction as \vec{a} is given by:

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

9.5 Dot Product

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}||\vec{b}|\cos\theta \\ \vec{a} \cdot \vec{b} &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

9.6 Properties of the Dot Product

Two vectors are orthogonal if their dot product is 0.

$$\begin{array}{ll} \vec{a} \cdot \vec{a} = |\vec{a}|^2 & \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \\ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} & (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b}) \\ 0 \cdot \vec{a} = 0 & \end{array}$$

9.7 Vector Projections

Scalar projection of \vec{b} onto \vec{a} :

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

Vector projection of \vec{b} onto \vec{a} :

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|}$$

9.8 Cross Product

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin \theta) \vec{n}$$

where \vec{n} is the unit vector orthogonal to both \vec{a} and \vec{b} .

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

9.9 Properties of the Cross Product

Two vectors are parallel if their cross product is 0.

$$\begin{aligned} \vec{a} \times \vec{b} &= -\vec{b} \times \vec{a} & (c\vec{a}) \times \vec{b} &= c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}) \\ \vec{a} \times (\vec{b} + \vec{c}) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} & (\vec{a} + \vec{b}) \times \vec{c} &= \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \end{aligned}$$

9.10 Scalar Triple Product

The volume of the parallelepiped determined by vectors \vec{a} , \vec{b} , and \vec{c} is the magnitude of their scalar triple product:

$$\begin{aligned} V &= |\vec{a} \cdot (\vec{b} \times \vec{c})| \\ \vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{c} \cdot (\vec{a} \times \vec{b}) \end{aligned}$$

9.11 Vector Equation of a Line

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

9.12 Symmetric Equations of a Line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where the vector $\vec{c} = \langle a, b, c \rangle$ is the direction of the line.

The symmetric equations for a line passing through the points (x_0, y_0, z_0) and (x_1, y_1, z_1) are given by:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

9.13 Segment of a Line

The line segment from \vec{r}_0 to \vec{r}_1 is given by:

$$\vec{r}(t) = (1 - t)\vec{r}_0 + t\vec{r}_1 \quad \text{for } 0 \leq t \leq 1$$

9.14 Vector Equation of a Plane

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

where \vec{n} is the vector orthogonal to every vector in the given plane and $\vec{r} - \vec{r}_0$ is the vector between any two points on the plane.

9.15 Scalar Equation of a Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is a point on the plane and $\langle a, b, c \rangle$ is the vector normal to the plane.

9.16 Distance Between Point and Plane

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

where P is a point, Σ is a plane, Q is a point on plane Σ , and \vec{n} is the vector orthogonal to the plane.

9.17 Distance Between Point and Line

$$d(P, L) = \frac{|\vec{PQ} \times \vec{u}|}{|\vec{u}|}$$

where P is a point in space, Q is a point on the line L , and \vec{u} is the direction of line.

9.18 Distance Between Line and Line

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

where P is a point on line L , Q is a point on line M , \vec{u} is the direction of line L , and \vec{v} is the direction of line M .

9.19 Distance Between Plane and Plane

$$d = \frac{|e - d|}{|\vec{n}|}$$

where \vec{n} is the vector orthogonal to both planes, e is the constant of one plane, and d is the constant of the other. The distance between non-parallel planes is 0.

9.20 Quadric Surfaces

$$\text{Ellipsoid: } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Elliptic Paraboloid: } \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{Hyperbolic Paraboloid: } \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\text{Cone: } \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{Hyperboloid of One Sheet: } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\text{Hyperboloid of Two Sheets: } -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

9.21 Cylindrical Coordinates

To convert from cylindrical to rectangular:

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

To convert from rectangular to cylindrical:

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z$$

9.22 Spherical Coordinates

To convert from spherical to rectangular:

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

To convert from rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2 \quad \tan \theta = \frac{y}{x} \quad \cos \phi = \frac{z}{\rho}$$

10 Vector Functions

10.1 Limit of a Vector Function

$$\lim_{t \rightarrow a} \vec{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

10.2 Derivative of a Vector Function

$$\begin{aligned} \frac{d\vec{r}}{dt} = \vec{r}'(t) &= \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\ \vec{r}'(t) &= \langle f'(t), g'(t), h'(t) \rangle \end{aligned}$$

10.3 Unit Tangent Vector

$$T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

10.4 Derivative Rules for Vector Functions

$$\begin{aligned} \frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] &= \vec{u}'(t) + \vec{v}'(t) \\ \frac{d}{dt}[c\vec{u}(t)] &= c\vec{u}'(t) \\ \frac{d}{dt}[f(t)\vec{u}(t)] &= f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \\ \frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] &= \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \\ \frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] &= \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \\ \frac{d}{dt}[\vec{u}(f(t))] &= f'(t)\vec{u}'(f(t)) \end{aligned}$$

10.5 Integral of a Vector Function

$$\int_a^b \vec{r}(t) dt = \left\langle \int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right\rangle$$

10.6 Arc Length of a Vector Function

$$L = \int_a^b |\vec{r}'(t)| dt$$

10.7 Curvature

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$
$$\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$
$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

10.8 Normal and Binormal Vectors

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

10.9 Velocity and Acceleration

$$\vec{v}(t) = \vec{r}'(t)$$
$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

10.10 Parametric Equations of Trajectory

$$x = (v_0 \cos \alpha)t \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

10.11 Tangential and Normal Components of Acceleration

$$\vec{a} = v'\vec{T} + \kappa v^2\vec{N}$$

10.12 Equations of a Parametric Surface

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

11 Partial Derivatives

11.1 Limit of $f(x, y)$

If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a path C_2 , then $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ does not exist.

11.2 Strategy to Determine if Limit Exists

1. Substitute in for x and y . If point is defined, limit exists. If not, continue.
2. Approach (x, y) from the x -axis by setting $y = 0$ and taking $\lim_{x \rightarrow a}$. Compare this result to approaching (x, y) from the y -axis by setting $x = 0$ and taking $\lim_{y \rightarrow a}$. If these results are different, then the limit does not exist. If results are the same, continue.
3. Approach (x, y) from any nonvertical line by setting $y = mx$ and taking $\lim_{x \rightarrow a}$. If this limit depends on the value of m , then the limit of the function does not exist. If not, continue.
4. Rewrite the function in cylindrical coordinates and take $\lim_{r \rightarrow a}$. If this limit does not exist, then the limit of the function does not exist.

11.3 Continuity

A function is continuous at (a, b) if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

11.4 Definition of Partial Derivative

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .

11.5 Notation of Partial Derivative

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f$$

11.6 Clairaut's Theorem

If the functions f_{xy} and f_{yx} are both continuous, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

11.7 Tangent Plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

11.8 The Chain Rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

11.9 Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

11.10 Gradient

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

11.11 Directional Derivative

$$D_{\vec{u}}f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

where $\vec{u} = \langle a, b \rangle$ is a unit vector.

11.12 Maximizing the Directional Derivative

The maximum value of the directional derivative $D_{\vec{u}}f(x)$ is $|\nabla f(x)|$ and it occurs when \vec{u} has the same direction as the gradient vector $\nabla f(x)$.

11.13 Second Derivative Test

Let $D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
3. If $D < 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is not a local maximum or minimum, but could be a saddle point.

11.14 Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

1. Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = k$$

2. Evaluate f at all of these points. The largest is the maximum value, and the smallest is the minimum value of f subject to the constraint g .

12 Multiple Integrals

12.1 Volume under a Surface

$$V = \iint_D f(x, y) \, dx \, dy$$

12.2 Average Value of a Function of Two Variables

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) \, dx \, dy$$

12.3 Fubini's Theorem

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

12.4 Splitting a Double Integral

$$\iint_R g(x)h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy$$

12.5 Double Integral in Polar Coordinates

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

12.6 Surface Area

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

where a smooth parametric surface S is given by $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$.

12.7 Surface Area of a Graph

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

12.8 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\theta d\phi$$

13 Vector Calculus

13.1 Line Integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

13.2 Fundamental Theorem of Line Integrals

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

13.3 Path Independence

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D .

13.4 Curl

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}$$

13.5 Conservative Vector Field Test

\vec{F} is conservative if $\text{curl} \vec{F} = 0$ and the domain is closed and simply connected.

13.6 Divergence

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F}$$

13.7 Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(\vec{F}) dx dy$$

13.8 Surface Integral

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$$

13.9 Flux

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$$

13.10 Stokes' Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

13.11 Divergence Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}(\vec{F}) dV$$

14 Appendix A: Selected Surface Parametrizations

14.1 Sphere of Radius ρ

$$\vec{r}(u, v) = \langle \rho \cos u \sin v, \rho \sin u \sin v, \rho \cos v \rangle$$

14.2 Graph of a Function $f(x, y)$

$$\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$$

14.3 Graph of a Function $f(\phi, r)$

$$\vec{r}(u, v) = \langle v \cos u, v \sin u, f(u, v) \rangle$$

14.4 Plane Containing $P, \vec{u},$ and \vec{v}

$$\vec{r}(s, t) = \vec{OP} + s\vec{u} + t\vec{v}$$

14.5 Surface of Revolution

$$\vec{r}(u, v) = \langle g(v) \cos u, g(v) \sin u, v \rangle$$

where $g(z)$ gives the distance from the z-axis.

14.6 Cylinder

$$\vec{r}(u, v) = \langle \cos u, \sin u, v \rangle$$

14.7 Cone

$$\vec{r}(u, v) = \langle v \cos u, v \sin u, v \rangle$$

14.8 Paraboloid

$$\vec{r}(u, v) = \langle \sqrt{v} \cos u, \sqrt{v} \sin u, v \rangle$$

15 Appendix B: Selected Differential Equations

15.1 Heat Equation

$$f_t = f_{xx}$$

15.2 Wave Equation (Waveequation)

$$f_{tt} = f_{xx}$$

15.3 Transport (Advection) Equation

$$f_x = f_t$$

15.4 Laplace Equation

$$f_{xx} = -f_{yy}$$

15.5 Burgers Equation

$$f_{xx} = f_t + f f_x$$