Harvard College

Math 21b: Linear Algebra and Differential Equations

Formula and Theorem Review

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Contents

1 Linear Equations

1.1 Standard Representation of a Vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

1.2 Reduced Row Echelon Form

- If a column contains a leading 1, then all ther other entires in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

1.3 Elementary Row Operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.

1.4 Rank of a Matrix

The rank of a matrix A is the number of leading 1's in $rref(A)$.

1.5 Dot Product of Vectors

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n$$

1.6 The Product $A\vec{x}$

$$A\vec{x} = \begin{bmatrix} \cdots & \vec{w}_1 & \cdots \\ & \vdots & \\ \cdots & \vec{w}_n & \cdots \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

1.7 Algebraic Rules for $A\vec{x}$

- $A(\vec{x}\vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$

2 Linear Transformations

2.1 Linear Transformations

A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if there exists a matrix A such that

$$T(\vec{x}) = A\vec{x}$$

A transformation is called linear if and only if

- $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
- $T(k\vec{v}) = kT(\vec{v})$

2.2 Scaling Matrix

A scaling matrix by k has the form:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

2.3 Orthogonal Projection onto a Line

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$$

where $\vec{w}$ is a vector parallel to the line L.

2.4 Reflection Matrix

A reflection matrix has the form:

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

2.5 Rotation Matrix

A rotation matrix has the form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2.6 Shear Matrix

A horizontal shear matrix has the form:

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and a vertical shear matrix has the form:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

2.7 Matrix Multiplication

The product of matrices BA is defined as the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$.

- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(kA)B = A(kB) = k(AB)$

2.8 Invertibility

An $n \times n$ matrix A is invertible if and only if:

1. $rref(A) = I_n$
2. $rank(A) = n$

or, more simply, if and only if $det(A) \neq 0$.

2.9 Finding the Inverse

1. Form the $n \times (2n)$ matrix $[A|I_n]$
2. Compute $rref[A|I_n]$
3. If $rref[A|I_n]$ is of the form $[I_n|B]$ then $A^{-1} = B$
4. If $rref[A|I_n]$ is of another form, then A is not invertible

2.10 Properties of Invertible Matrices

If an $n \times n$ matrix A is invertible:

- The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for all $\vec{b}$ in $\mathbb{R}^n$
- $rref(A) = I_n$
- $rank(A) = n$
- $im(A) = \mathbb{R}^n$
- $ker(A) = \vec{0}$
- The column vectors of A are linearly independent and form a basis of $\mathbb{R}^n$

3 Subspaces of $\mathbb{R}^n$ and Their Dimensions

3.1 Image of a Function

$$image(f) = \{f(x) : x \text{ in } X\} = \{b \text{ in } Y : b = f(x), \text{ for some } x \text{ in } X\}$$

For a linear transformation T:

- The zero vector is in the image of T
- If $\vec{v}_1$ and $\vec{v}_2$ are in the image of T, then so is $\vec{v}_1 + \vec{v}_2$
- If $\vec{v}$ is in the image of T, then so is $k\vec{v}$

3.2 Span

$$span(\vec{v}_1, \cdots, \vec{v}_n) = \{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m : c_1 \cdots c_n \text{ in } \mathbb{R}\}$$

3.3 Kernel

The kernel of a linear transformation T is the solution set of the linear system:

$$A\vec{x} = \vec{0}$$

- The zero vector is in the kernel of T
- If $\vec{v}_1$ and $\vec{v}_2$ are in the kerne; of T, then so is $\vec{v}_1 + \vec{v}_2$
- If $\vec{v}$ is in the kernel of T, then so is $k\vec{v}$

3.4 Subspaces of $\mathbb{R}^n$

A subset W of the vector space $\mathbb{R}^n$ is a linear subspace if and only if:

1. W contains the zero vector
2. W is closed under addition
3. W is closed under scalar multiplication

The image and kernel of a linear transformation are linear subspaces.

3.5 Linear Independence

1. If a vector $\vec{v}_i$ in $\vec{v}_1, \cdots, \vec{v}_n$ can be expressed as a linear combination of the vectors $\vec{v}_1, \cdots, \vec{v}_{i-1}$, then $\vec{v}_i$ is redundant.
2. The vectors $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent if no vector is redundant.
3. The vectors $\vec{v}_1, \cdots, \vec{v}_n$ form a basis of a subspace V if they span V and are linearly independent

3.6 Dimension

The number of vectors in a subspace V is the dimension of V.

3.7 Rank-Nullity Theorem

For an $n \times m$ matrix A:

$$dim(kerA) + dim(imA) = m$$

3.8 Coordinates

For a basis $\mathfrak{B} = (\vec{v}_1, \cdots, \vec{v}_n)$ of a subspace V, any vector $\vec{x}$ can be written as:

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

$c_1, \cdots, c_n$ are called the $\mathfrak{B}$-coordinates of $\vec{x}$, with

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}} \quad \text{and} \quad [\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x}$$

where $S = [v_1 \cdots v_n]$

3.9 Linearity of Coordinates

- $[\vec{x} + \vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}}$
- $[k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}}$

3.10 Matrix of a Linear Transformation

The matrix B that transforms $[\vec{x}]_{\mathfrak{B}}$ into $[T(\vec{x})]_{\mathfrak{B}}$ is called the $\mathfrak{B}$-matrix of T:

$$[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}, \quad \text{where } B = [\,[T(\vec{v}_1)]_{\mathfrak{B}} \cdots [T(\vec{v}_n)]_{\mathfrak{B}}\,]$$

3.11 Similar Matrices

Two matrices A and B are similar if and only if:

$$AS = SB \quad \text{or} \quad B = S^{-1}AS$$

4 Linear Spaces

4.1 Linear Spaces

A linear space V is a set with rules for addition and scalar multiplication that satisfies the following properties:

1. $(f+g)+h = f+(g+h)$
2. $f+g = g+f$
3. There exists a neutral element n in V such that $f+n = f$
4. For each f in V there exists a g in V such that $f+g = 0$
5. $k(f+g) = kf+kg$
6. $(c+k)f = cf+kf$
7. $c(kf) = (ck)f$
8. $1f = f$

4.2 Finding a Basis of a Linear Space V

1. Find a typical element w of V in terms of arbitrary constants.
2. Express w as a linear combination of elements in V.
3. If these elements are linearly independent, they will for a basis of V.

4.3 Isomorphisms

- A linear transformation T is an isomorphism of T is invertible.
- A linear transformation T from V to W is an isomorphism if any only if $ker(T) = 0$ and $im(T) = W$.
- Coordinate changes are isomorphisms.
- If V is isomorphic to W, then $dim(V) = dim(W)$.

5 Orthogonality and Least Squares

5.1 Orthongality and Length

- Two vectors $\vec{c}$ and $\vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.
- The length of a vector $\vec{v}$ is $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$.
- A vector $\vec{u}$ is a unit vector if its length is 1.

5.2 Orthonormal Vectors

The vectors $\vec{u}_1, \cdots, u_n$ are orthonormal if they are unit vectors orthogonal to each other.

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

5.3 Orthogonal Projection onto a Subspace

If V is a subspace with an orthonormal basis $\vec{u}_1, \cdots, \vec{u}_n$:

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$$

5.4 Orthogonal Complement

The orthogonal complement $V^\perp$ of a subspace V is the set of vectors $\vec{x}$ orthogonal to all vectors in V:

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \text{ for all } \vec{v} \text{ in } V\}$$

- $V^\perp$ is a subspace of V
- $V \cap V^\perp = \vec{0}$
- $dim(V) + dim(V^\perp) = n$
- $(V^\perp)^\perp = V$

5.5 Pythongorean Theorem

$$||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$$

5.6 Cauchy-Schwarz Inequality

$$||\vec{x} \cdot \vec{y}|| \leq ||\vec{x}||||\vec{y}||$$

5.7 Angle Between Two Vectors

$$\theta = \arccos \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||||\vec{y}||}$$

5.8 Gram-Schmidt Process

For a basis $\vec{v}_1, \cdots, \vec{v}_n$ of a subspace V:

$$\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1, \cdots, \vec{u}_n = \frac{1}{||\vec{v}_n^{\perp}||}$$

where

$$\vec{v}_i^{\perp} = \vec{v}_i - (\vec{u}_1 \cdot v_i)\vec{u}_1 - \cdots - (\vec{u}_{i-1} \cdot v_i)\vec{v}_{i-1}$$

5.9 QR Decomposition

For an $n \times m$ matrix M, $M = QR$, where Q is an $n \times m$ matrix whose columns $\vec{u}_1, \cdots, \vec{u}_n$ are orthonormal and R has entries satisfying:

$$r_1 1 = ||\vec{v}_1||, \quad r_{jj} = ||\vec{v}_j^{\perp}||, \quad r_{ij} = \vec{u}_i \cdot \vec{v}_j \text{ for } i < j$$

5.10 Orthongonal Transformation

A linear transformation T is considered orthogonal if it preserves the length of vectors, such that

$$||T(\vec{x})|| = ||\vec{x}||$$

- T is orthogonal if the vectors $T(\vec{e}_1), \cdots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$.
- The matrix A is orthogonal if $A^T A = I_n$.
- The matrix A is orthogonal if $A^{-1} = A^T$.
- A matrix A is orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$.
- The product AB of two orthogonal matrices A and B is orthogonal.
- The inverse A^{-1} of an orthogonal matrix A is orthogonal.

5.11 Transpose

The transpose A^T of an $m \times n$ matrix A is the $n \times m$ matrix whose ijth entry is the jith entry of A.

- $(AB)^T = B^T A^T$
- $(A^T)^{-1} = (A^{-1})^T$
- $rank(A) = rank(A^T)$.

5.12 Symmetric and Skew Symmetric Matrices

- An $n \times n$ matrix A is symmetric if $A^T = A$.
- An $n \times n$ matrix A is skew-symmetric if $A^T = -A$.

5.13 Matrix of an Orthogonal Projection

The orthongonal projection onto a subspace V with an orthonormal basis $\vec{u}_1, \cdots, \vec{u}_n$ is

$$QQ^T, \quad \text{where } Q = [\vec{u}_1 \cdots \vec{u}_n]$$

or equivalently,

$$A(A^TA)^{-1}A^T \quad \text{where } A = [\vec{v}_1 \cdots \vec{v}_n]$$

5.14 Least-Squares Solution

The unique least-squares solution of a linear system $A\vec{x} = \vec{b}$ where $ker(A) = \vec{0}$ is

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

5.15 Inner Product Spaces

The inner product of a linear space V, denoted $\langle f, g\rangle$, has the following properties:

- $\langle f, g\rangle = \langle g, f\rangle$
- $\langle f + h, g\rangle = \langle f, g\rangle + \langle h, g\rangle$
- $\langle cf, g\rangle = c\langle f, g\rangle$
- $\langle f, f\rangle > 0$

5.16 Norm and Orthongonality

The norm of an element f of an inner product space is

$$||f|| = \sqrt{\langle f, f\rangle}$$

Two elements f and g of an inner product space are orthongonal if

$$\langle f, g\rangle = 0$$

5.17 Trace of a Matrix

The trace $tr(A)$ of a matrix A is the sum of its diagonal entries.

5.18 Orthogonal Projection of an Inner Product Space

If $g_1, \cdots, g_n$ is an orthonormal basis of a subspace W of an inner product space V:

$$\text{proj}_W f = \langle g_1, f \rangle g_1 + \cdots + \langle g_n, f \rangle g_n$$

5.19 Fourier Analysis

$$f_n(t) = a_0 \frac{1}{\sqrt{2}} + b_1 \sin(t) + c_1 \cos(t) + \cdots + b_n \sin(nt) + c_n \cos(nt)$$

where

$$a_0 = \langle f, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} f(t)\ dt$$

$$b_k = \langle f, \sin(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\ dt$$

$$c_k = \langle f, \cos(kt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\ dt$$

6 Determinants

6.1 Sarrus's Rule

For an $n \times n$ matrix A, write the first $n-1$ columns to the right of A, then multiply along the diagonal to get $2n$ products. Subtract the first n products, then add the second n products to get the determinant.

6.2 Patterns

- A pattern of an $n \times n$ matrix A is a way to choose n entries of A such that each entry is in a unique row and column.
- The product of a pattern is designated P.
- Two entries in a pattern are an inversion if one is located above and to the right of the other.
- The signature of a pattern is defined as $sgn\ P = (-1)$(inversions in P)

$$det(A) = \sum (sgn\ P)(prod\ P)$$

6.3 Determinants and Gauss-Jordan Elimination

If B is an $n \times n$ matrix obtained from applying an elementary row operation on an $n \times n$ matrix A:

- If B is obtained by row division: $det(B) = \frac{1}{k}det(A)$
- If B is obtained by row swap: $det(B) = -det(A)$
- If B is obtained by row addition: $det(B) = det(A)$

6.4 Laplace Expansion

Expansion down the jth column:

$$det(A) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}det(A_{ij})$$

Expansion along the ith row:

$$det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}det(A_{ij})$$

6.5 Properties of the Determinant

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- If A and B are similar, then $det(A) = det(B)$
- $det(A^{-1}) = \frac{1}{det(A)} = det(A)^{-1}$

7 Eigenvalues and Eigenvectors

7.1 Eigenvalues

λ is an eigenvalues of an $n \times n$ matrix A if and only if

$$det(A - \lambda I_n) = 0$$

7.2 Characteristic Polynomial

$$det(A - \lambda I_n) = (-1)^n\lambda^n + (-1)^{n-1}tr(A)\lambda^{n-1} + \cdots + det(A)$$

7.3 Algebraic Multiplicity

An eigenvalue λ has algebraic multiplicty k if it is a root of multiplicity k of the characteristic polynomial.

7.4 Eigenvalues, Determinant, and Trace

For an $n \times n$ matrix A

$$det(A) = \lambda_1 \cdots \lambda_n = \prod_{k=1}^{n} \lambda_k$$

$$tr(A) = \lambda_1 + \cdots + \lambda_n = \sum_{k=1}^{n} \lambda_k$$

7.5 Eigenspace

$$E_\lambda = ker(A - \lambda I_n) = \{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

7.6 Eigenbasis

An eigenbasis for an $n \times n$ matrix A conists of the eigenvectors of A and forms a basis for $\mathbb{R}^n$.

7.7 Geometric Multiplicty

The dimension of the eignspace E_λ is the geometric multiplicty of the eigenvalue λ.

7.8 Diagonalization

1. Find the eigenvalues and corresponding eigenvectors of the matrix A.
2. Let S be the eigenbasis for A and D be a matrix with the eigenvalues of A along the diagonal.
3. $D = S^{-1}AS$ and $A = SDS^{-1}$

7.9 Powers of a Diagonalizable Matrix

If $A = SDS^{-1}$, then

$$A^t = SD^tS^{-1}$$

7.10 Stable Equilibrium

$\vec{0}$ is an asympototically stable equilibrium for the system $\vec{x}(t+1) = A\vec{x}(t)$ if

$$\lim_{t\to\infty} \vec{x}(t) = \lim_{t\to\infty} A^t = \vec{0}$$

9 Linear Differential Equations

9.1 Exponential Growth and Decay

$$\frac{dx}{dt} = kx, \quad x(t) = e^{kt}x_0$$

9.2 Linear Dynamical Systems

- Discrete model: $\vec{x}(t+1) = B\vec{x}(t)$
- Continuous model: $\frac{d\vec{x}}{dt} = A\vec{x}$

9.3 Continuous Dynamical Systems with Real Eigenvalues

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

where $\vec{v}_1, \cdots \vec{v}_n$ forms a real eigenbasis of A with eignvalues $\lambda_1, \cdots, \lambda_n$

9.4 Continuous Dynamical Systems with Complex Eigenvalues

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(t) = e^{pt} S \begin{bmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{bmatrix} S^{-1}\vec{x}_0$$

where $p \pm q$ are eigenvalues with eigenvectors $\vec{v} \pm \vec{w}$ and $S = [\vec{w}\vec{v}]$.

9.5 Strategy for Solving Linear Differential Equations

To solve an nth order linear differential equation with the form $T(f) = g$:

1. Find a basis $f_1, \cdots, f_n$ of $ker(T)$.
2. Find a particular solution f_p.
3. The solutions f are in the form $f = c_1 f_1 + \cdots + c_n f_n + f_p$.

9.6 Eigenfunctions

A smooth function F is an eigenfunction of T if

$$T(f) = \lambda f$$

9.7 Characteristic Polynomial of a Linear Differential Operator

For $T(f) = f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1 f + a_0 f$, the characteristic polynomial is defined as:

$$p_T(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

If $e^{\lambda t}$ is an eigenfunction of T with eigenvalue $p_T(\lambda)$:

$$T(e^{\lambda t}) = p_T(\lambda)e^{\lambda t}$$

9.8 Kernel of a Linear Differential Operator

If T is a linear differential operator with characteristic polynomial $p_T(\lambda)$ with roots $\lambda_1, \cdots, \lambda_n$, then the kernel of T is formed by

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t}$$

9.9 Characteristic Polynomial with Complex Solution

If the zeros of $p_T(\lambda)$ are $p \pm q$, then the solutions to its differential equation are of the form

$$f(t) = e^{pt}(c_1 \cos(qt) + c_2 \sin(qt))$$

9.10 First-Order Linear Differential Equations

A differential equation of the form $f'(t) - af(t) = g(t)$ has a solution of the form

$$f(t) = e^{at} \int e^{-at} g(t)\ dt$$

9.11 Strategy for Solving Linear Differential Equations

To solve the nth order linear differential equation $T(f) = g$:

1. Find n linearly independent solutions of $T(f) = 0$.
   - Write the characteristic polynomial $p_T(\lambda)$ of T by replacing $f^{(k)}$ with λ^k
   - Find the solutions $\lambda_1, \cdots, \lambda_n$ of the equation $p_T(\lambda) = 0$.
   - If λ is a solution of $p_T(\lambda) = 0$, then $e^{\lambda t}$ is a solution of $T(f) = 0$.

- If λ is a solution of $p_t(\lambda) = 0$ with multiplicity m, then $e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \cdots, t^{m-1}e^{\lambda t}$ are the solutions of $T(f) = 0$.
- If $p \pm q$ are complex solutions of $p_T(\lambda) = 0$, then $e^{pt}\cos(qt)$ and $e^{pt}\sin(qt)$ are real solutions of $T(f) = 0$.

2. If $T(f)$ is inhomogenous, find one particular solution f_p of the equation $T(f) = g$.
   - If g is of the form $g(t) = A\cos(\omega t) + B\sin(\omega t)$, $g(t) = A\cos(\omega t)$, or $g(t) = A\sin(\omega t)$, look for a particular solution of the form $f_p(t) = P\cos(\omega t) + Q\sin(\omega t)$.
   - If g is of the form $g(t) = a_0 + a_1 t + \cdots a_n t^n$, look for a particular solution of the form $f_p(t) = c_0 + c_1 t + \cdots c_n t^n$
   - If g is constant, look for a particular solution of the form $f_p(t) = c$.
   - If g is of the form $g(t) = f'(t) - af(t)$, use the formula $f(t) = e^{at}\int e^{-at}g(t)\ dt$.
3. The solutions of $T(f) = g$ are of the form

$$f(t) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) + f_p(t)$$

where $f_1, \cdots, f_n$ are the solutions from Step 1 and f_p is the solution from Step 2.

9.12 Linearization of a Nonlinear System

If (a, b) is an equilibrium point of the system $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, such that $f(a, b) = 0$ and $g(a, b) = 0$, then the system is approximated near (a, b) by the Jacobian matrix:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

9.13 The Heat Equation

$$f_t(x,t) = \mu f_{xx}(x,t)$$

has solutions of the form

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2\mu t} \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x,0)\sin(nx)\ dx$$

9.14 The Wave Equation

$$f_{tt}(x,t) = c^2 f_{xx}(x,t)$$

has solutions of the form

$$f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x,0)\sin(nx)\ dx$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f_t(x,0)\sin(nx)\ dx$.