Malden Catholic High School

Advanced Placement* Calculus

Formula and Theorem Review

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May 3, 2009

**This review is not affiliated with the College Board or AP program.*

Contents

1 Prerequisites for Calculus

1.1 Increments

$$\Delta x = x_2 - x_1$$

1.2 Slope

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta y} = \frac{y_2 - y_1}{x_2 - x_1}$$

1.3 Point-Slope Equation

$$y - y_1 = m(x - x_1)$$

1.4 Slope-Intercept Equation

$$y = mx + b$$

1.5 General Linear Equation

$$Ax + By = C$$

1.6 Properties of Logarithms

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
$$\log_a x^y = y \log_a x$$

1.7 Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}$$

2 Limits and Continuity

2.1 Properties of Limits

$$\lim_{x \to c}(f(x) + g(x)) = L + M$$
$$\lim_{x \to c}(f(x) - g(x)) = L - M$$
$$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$$
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$$
$$\lim_{x \to c}(k \cdot f(x)) = k \cdot L$$
$$\lim_{x \to c}(f(x))^{\frac{r}{s}} = L^{\frac{r}{s}}$$

2.2 Sandwich Theorem

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then $\lim_{x \to c} f(x) = L$

3 Derivatives

3.1 Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3.2 Derivative at a Point

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3.3 Derivative of a Constant Function

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

3.4 Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

3.5 Constant Multiple Rule

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

3.6 Sum and Difference Rule

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

3.7 Product Rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

3.8 Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

3.9 Velocity and Acceleration

$$\begin{aligned} v(t) &= \frac{ds}{dt} \\ a(t) &= \frac{dv}{dt} = \frac{d^2s}{dt^2} \end{aligned}$$

3.10 Derivatives of Trigonometric Functions

$$\begin{array}{ll} \frac{d}{dx}\sin x = \cos x & \frac{d}{dx}\cos x = -\sin x \\ \frac{d}{dx}\tan x = \sec^2 x & \frac{d}{dx}\sec x = \sec x \tan x \\ \frac{d}{dx}\csc x = -\csc x \cot x & \frac{d}{dx}\cot x = -\csc^2 x \end{array}$$

3.11 Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

3.12 Parametric Derivative

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

3.13 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$
$$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$$
$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$

3.14 Function-Cofunction Inverse Identities

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$
$$\csc^{-1}x = \frac{\pi}{2} - \cos^{-1}x$$
$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

3.15 Calculator Conversion Identities

$$\sec^{-1}x = \cos^{-1}\frac{1}{x}$$
$$\csc^{-1}x = \sin^{-1}\frac{1}{x}$$
$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

3.16 Derivative of a^x

$$\frac{d}{dx}\,a^u = a^u\ln a\frac{du}{dx}$$

3.17 Derivative of $\ln x$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

3.18 Derivative of $\log_a x$

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$$

4 Applications of Derivatives

4.1 Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

1. Absolute maximum if $f(x) \leq f(c)$ for all x in D.

2. Absolute minimum if $f(x) \geq f(c)$ for all x in D.

4.2 Extreme Value Theorem

If f is continuous and differentiable on $[a, b]$ then f has both a maximum value and a minimum value on the interval.

4.3 Candidates for Local Extrema

1. Left end of function

2. Right end of function

3. $f'(c) = 0$

4. $f'(c) =$ undefined

4.4 Mean Value Theorem

If $y = f(x)$ is continuous and differentiable for every point on the closed interval $[a, b]$, then there is at least one point c on the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4.5 Increasing and Decreasing Functions

1. f is increasing if $x_1 < x_2$ and $f(x_1) < f(x_2)$ or $f' > 0$

2. f is decreasing if $x_1 < x_2$ and $f(x_1) > f(x_2)$ or $f' < 0$

4.6 First Derivative Test for Local Extrema

1. If f' changes sign from positive to negative at c then f has a local maximum at c

2. If f' changes sign from negative to positive at c then f has a local minimum at c

3. If f' does not change sign at c then f has no extreme value at c

4.7 Concavity Test

1. If $f''(x) > 0$ then $f(x)$ is concave up
2. If $f''(x) < 0$ then $f(x)$ is concave down
3. If $f''(x) = 0$ then $f(x)$ is a candidate for a point of inflection

4.8 Second Derivative Test for Local Extrema

1. If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at $x = c$
2. If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at $x = c$

4.9 First and Second Derivative Summary

$y' > 0$	y is increasing
$y' < 0$	y is decreasing
$y' = 0$	y is possible extrema
$y' = \infty$	y is possible extrema
$y'' > 0$	y is concave up, minimum if $y' = 0$
$y'' < 0$	y is concave down, maximum if $y' = 0$
$y'' = 0$	y is possible point of inflection

4.10 Economics

If $r(x)$ is the revenue of selling x items and $c(x)$ is the cost of producing x items, then

$$p(x) = r(x) - c(x)$$

$r'(x) =$ Marginal Revenue and $c'(x) =$ Marginal Cost
Maximum profit occurs where $r'(x) - c'(x) = 0$

$$\text{Average cost} = \frac{c(x)}{x}$$

4.11 Linearization of $f(x)$ about $x = a$

$$L(x) = f(a) + f'(a)(x - a)$$

4.12 Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

4.13 Steps for Solving Related Rates Problems

1. Draw a sketch.
2. Write down known and unknown variables.
3. Write an equation relating the known and unknown variables. Make sure units are consistent.
4. Differentiate implicitly with respect to time.
5. Solve for the unknown variable.

5 The Definite Integral

5.1 The Definite Integral as a Limit of Riemann Sum

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

5.2 Area Under a Curve

$$A = \int_a^b f(x)\ dx$$

5.3 Integral of a Constant

$$\int_a^b f(x)\ dx = \int_a^b c \cdot dx = c\,(b-a)$$

5.4 Properties of Definite Integrals

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$
$$\int_a^a f(x)\ dx = 0$$
$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$$
$$\int_a^b -f(x)\ dx = -\int_a^b f(x)\ dx$$
$$\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$$
$$\int_a^b f(x)\ dx + \int_b^c f(x)\ dx = \int_a^c f(x)\ dx$$

5.5 Max-Min Inequality

$$\mathrm{min} f \cdot (b-a) \leq \int_a^b f(x)\ dx \leq \mathrm{max} f \cdot (b-a)$$

5.6 Average (Mean) Value

If f is integrable on $[a,b]$, its average value on $[a,b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

5.7 Mean Value Theorem for Definite Integrals

If $y = f(x)$ is continuous and differentiable for every point on the closed interval $[a,b]$, then there is at least one point c on the open interval (a,b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

5.8 The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^b f(t)\ dt$$

has a derivative on every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

5.9 The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

5.10 Trapezoidal Rule

To approximate $\int_a^b f(x)\ dx$, use

$$T = \frac{b-a}{2n}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

where $[a, b]$ is partitioned into n subintervals of equal length
This is also given as the average of LRAM and RRAM.

$$T = \frac{LRAM_n + RRAM_n}{2}$$

5.11 Simpson's Rule

To approximate $\int_a^b f(x)\ dx$, use

$$S = \frac{b-a}{3n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

where $[a, b]$ is partitioned into an even number of n subintervals of equal length

5.12 Error Bounds for Trapezoidal and Simpson's Rule

If T and S represent the approximation of $\int_a^b f(x)\ dx$ given by the Trapezoidal Rule and Simpson's Rule, respectively, then the errors E_T and E_S satisfy

$$|E_T| \leq \tfrac{b-a}{12}h^2 M_{f''} \quad \text{and} \quad |E_S| \leq \tfrac{b-a}{180}h^4 M_{f^{IV}}$$

6 Differential Equations and Mathematical Modeling

6.1 Euler's Method

1. Begin at the point (x, y) specified by the initial condition.
2. Use the differential equation to find $\frac{dy}{dx}$.
3. Increase x by a small dx.
4. Increase y by $dy = \frac{dy}{dx} \cdot dx$.
5. Calculate the next point, given by $(x + dx, y + dy)$.
6. Use this point as (x, y) and repeat the process.

Table set up:

(x, y)	$\frac{dy}{dx}$	dx	$dy = \frac{dy}{dx} \cdot dx$	$(x + dx, y + dy)$

6.2 Integration by Substitution

1. Let u equal some function within the integrand.
2. Find $\frac{du}{dx}$.
3. Solve for dx in terms of du.
4. Substitute u into the integrand.
5. Substitute the function obtained in Step 3 for dx.
6. Pull out any constants.
7. Integrate with respect to $u(a)$ and $u(b)$.

6.3 Integration by Parts

$$\int u \; dv = uv - \int v \; du$$

6.4 Separable Differential Equation

A differential equation in the form $\frac{dy}{dx} = f(y)\; g(x)$ can be separated by writing it in the form:

$$\frac{1}{f(y)} \; dy = g(x) \; dx$$

6.5 Law of Exponential Change

If y changes at a rate proportional to the amount present, as in the function $\frac{dy}{dt} = ky$, and $y = y_0$ when $t = 0$, then

$$y = y_0 \cdot e^{kt}$$

6.6 Compounded Interest

$$A(t) = A_0 \left(1 + \frac{r}{k}\right)^{kt}$$

where A_0 represents the initial deposit, r represents the interest rate, k represents the number of times compounded per year, and t represents the number of years.

6.7 Half-Life

$$y = y_0 \cdot 2^{-t/t_{1/2}}$$

where y_0 represents the initial amount present, t represents time, and $t_{1/2}$ represents half-life.

6.8 Newton's Law of Cooling

$$\frac{T - T_s}{T_0 - T_s} = e^{-kt}$$

6.9 Logistic Differential Equation

The logistic differential equation in the form:

$$\frac{dP}{dt} = kP(M - P)$$

where P represents the population and M represents the maximum capacity has the solution:

$$P = \frac{M}{1 + Ae^{-Mkt}}$$

where A is given by $\frac{M-P_0}{P_0}$

7 Applications of Definite Integrals

7.1 Area Between Curves

If f and g are continuous with $f(x) > g(x)$ then the area between the curves is:

$$A = \int_a^b [f(x) - g(x)]\ dx$$

7.2 Volume of a Solid

$$V = \int_a^b A(x)\ dx$$

where $A(x)$ represents cross-sectional area from $x = a$ to $x = b$

7.3 Volume by Method of Slicing

1. Sketch the solid and a typical cross-section
2. Find a formula for $A(x)$ or $A(y)$ and the appropriate limits of integration
3. Ensure the variable expressed in the area function matches the differential term
4. Integrate to find volume with the appropriate units

7.4 Volume by Method of Cylindrical Shells

$$\int_a^b 2\pi \cdot x \cdot f(x) \quad \text{or} \quad \int_a^b 2\pi \cdot y \cdot f(y)$$

7.5 Area of a Washer

$$A = \pi(R^2 - r^2)$$

where R is the outer radius of the washer and r is the inner radius of the washer

7.6 Surface Area

$$\text{About the x-axis: } SA = \int_a^b 2\pi \cdot f(x) \cdot \sqrt{1 + f'(x)^2}\ dx$$

$$\text{About the y-axis: } SA = \int_a^b 2\pi \cdot f(y) \cdot \sqrt{1 + f'(y)^2}\ dy$$

7.7 Cavalieri's Theorem

Solids with equal altitudes and identical cross-sectional areas at each height have the same volume.

7.8 Length of a Smooth Curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \quad \text{or} \quad L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

7.9 Work

$$W = \int_a^b F(x)\ dx$$

7.10 Work Done Pumping

$$W = \int_{\text{bottom depth}}^{\text{top depth}} \text{density} \cdot \text{height} \cdot \text{cross-sectional area} \cdot dx$$

where height and cross-sectional area are functions of the same variable.

7.11 Normal Probability Distribution Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

8 Sequences, L'Hopital's Rule, and Improper Integrals

8.1 Arithmetic Sequence

$$\begin{aligned} \textit{Recursive} &: a_n = a_{n-1} + d \\ \textit{Explicit} &: a_n = a_1 + (n-1)d \end{aligned}$$

8.2 Geometric Sequence

$$\begin{aligned} \textit{Recursive} &: a_n = a_{n-1} \cdot r \\ \textit{Explicit} &: a_n = a_1 \cdot r^{n-1} \end{aligned}$$

8.3 L'Hopital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

8.4 Indeterminate Forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

8.5 Relative Rates of Growth

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

2. g grows faster than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

3. f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$$

8.6 Improper Integrals with Infinite Integration Limits

1. if $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

2. if $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$$

3. if $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^c f(x)\ dx + \int_c^\infty f(x)\ dx$$

8.7 Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

1. $\int_a^\infty f(x)\ dx$ converges if $\int_a^\infty g(x)\ dx$ converges.
2. $\int_a^\infty g(x)\ dx$ diverges if $\int_a^\infty f(x)\ dx$ diverges.

9 Infinite Series

9.1 Convergence of a Geometric Series

The infinite series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

converges to the sum

$$\frac{a_1}{1-r}$$

if $|r| < 1$

9.2 Partial Sum of a Geometric Series

$$S_n = a_1 \cdot \frac{1-r^n}{1-r}$$

9.3 Term-by-Term Differentiation

If

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

converges for $|x-a| < R$, then the series

$$g(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots$$

obtained by differentiating the series for f term by term, converges for $|x-a| < R$ and represents $f'(x)$ on that interval. If the series for f converges for all x, then so does the series for f'.

9.4 Term-by-Term Integration

If

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

converges for $|x-a| < R$, then the series

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} = c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + c_n\frac{(x-a)^{n+1}}{n+1} + \cdots$$

obtained by integrating the series for f term by term, converges for $|x-a| < R$ and represents $\int_a^x f(t)\,dt$ on that interval. If the series for f converges for all x, then so does the series for the integral.

9.5 Maclaurin Series

Let f be a function with derivatives of all orders throughout some open interval containing 0. The Maclaurin series generated by f is

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

9.6 Taylor Series

Let f be a function with derivatives of all orders throughout some open interval containing a. The Taylor series generated by f at $x = a$ is

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

9.7 Common Maclaurin Series

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1) \\
\frac{1}{1+x} &= 1 - x + x^2 - \cdots + (-x)^n \sum_{n=0}^{\infty} (-1)^n x^n \qquad (|x| < 1) \\
e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(all real } x\text{)} \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{(all real } x\text{)} \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{(all real } x\text{)} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad (-1 < x \leq 1) \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (|x| \leq 1)
\end{aligned}$$

9.8 Lagrange Form of the Remainder of a Taylor Polynomial

The remainder of partial sum S_n where c is between a and x is given by:

$$R_n(x) = |P_n(x) - f(x)| = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$$

9.9 Remainder Estimation Theorem

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where M is the maximum value of the $(n+1)$th derivative on the given interval.

9.10 Convergence Theorem for Power Series

There are three possibilities for $\sum_{n=0}^{\infty} c_n(x-a)^n$ with respect to convergence:

1. There is a positive number R such that the series diverges for $|x-a| > R$ but converges for $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a-R$ and $x = a+R$. R is known as the radius of convergence.

2. The series converges for every x ($R = \infty$)

3. The series converges at $x = a$ and diverges elsewhere ($R = 0$)

9.11 The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ failes to exist or is not zero.

9.12 Direct Comparison Test

Let $\sum a_n$ be a series with no negative terms:

1. $\sum a_n$ converges if there is some convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N. The geometric series $\sum_{n=0}^{\infty} ar^n$ and p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ can be used for convergence tests.

2. $\sum a_n$ diverges if there is some divergent series $\sum d_n$ of nonnegative terms with $a_n \geq d_n$ for all $n > N$, for some integer N. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ can be used for divergence tests.

9.13 Absolute Convergence

If the series $\sum |a_n|$ converges, then $\sum a_n$ converges.

9.14 Ratio Test

Let $\sum a_n$ be a series with positive terms and with

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

1. The series converges if $L < 1$
2. The series diverges if $L > 1$
3. The test is inconclusive if $L = 1$

9.15 Telescoping Series

$$\sum_{n=1}^{\infty} b_n - b_{n+1}$$

converges to $b_1 - \lim_{n\to\infty} b_{n+1}$

9.16 nth Root Test

Let $\sum a_n$ be a series with positive terms and with

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = L$$

1. The series converges if $L < 1$
2. The series diverges if $L > 1$
3. The test is inconclusive if $L = 1$

9.17 Integral Test

If a_n is a sequence of positive terms and $a_n = f(n)$ where $f(n)$ is a continuous, positive, decreasing function, then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\, dx$ both converge or both diverge.

9.18 Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$, where N is a positive integer.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c,\ 0 < c < \infty$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

9.19 Testing for Convergence of a Series

1. Is $\lim_{n\to\infty} a_n = 0$?
   (a) The series converges if so.
   (b) Test is inconclusive if not.
2. Is the series a geometric series in the form $\sum_{n=1}^{\infty} ar^n$?
   (a) The series converges if $r < 1$
   (b) The series diverges if $r > 1$
3. Is the series a p-series in the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$?
   (a) The series converges if $p > 1$
   (b) The series diverges if $p \leq 1$
4. Is the series an alternating series?
   (a) The series converges if $\lim_{n\to\infty} a_n = 0$ and the absolute values of the terms of the series are decreasing.
5. Is the series a telescoping series given in the form $\sum_{n=1}^{\infty} b_n - b_{n+1}$?
   (a) The series converges to $b_1 - \lim_{n\to\infty} b_{n+1}$
6. Is $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ easily expressed?
   (a) The series converges if $L < 1$
   (b) The series diverges if $L > 1$
   (c) The test is inconclusive if $L = 1$
7. Is a_n easily expressed as an integrable function?
   (a) The series converges if integral converges.
   (b) The series diverges if integral diverges.
8. Can the series easily be compared to a series of known convergence or divergence?
   (a) The series converges if series is less than series of known convergence.
   (b) The series diverges if series is greater than series of known convergence.
9. Is $\lim_{n\to\infty} (a_n)^{\frac{1}{n}}$ easily expressed?
   (a) The series converges if $L < 1$
   (b) The series diverges if $L > 1$
   (c) The test is inconclusive if $L = 1$

10 Parametric, Vector, and Polar Functions

10.1 Parametric Differentiation Formulas

If x and y are both differentiable functions of t and $\frac{dx}{dt} \neq 0$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

10.2 Length of a Paramatrized Curve

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

10.3 Surface Area of a Parametric Curve

$$\text{Revolved about the x-axis: } SA = \int_a^b 2\pi \cdot y(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\text{Revolved about the y-axis: } SA = \int_a^b 2\pi \cdot x(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

10.4 Properties of Vectors

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$
$$k\mathbf{u} = \langle ku_1, ku_2 \rangle$$

10.5 Vector Magnitude

$$|\mathbf{v}| = \sqrt{{v_1}^2 + {v_2}^2}$$

10.6 Angle Between Two Vectors

$$\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \cos^{-1} \frac{u_1 v_1 + u_2 v_2}{|\mathbf{u}||\mathbf{v}|}$$

10.7 Vector Dot Product

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = u_1 v_1 + u_2 v_2$$

10.8 Direction Vector

$$\text{Direction Vector} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

10.9 Polar-Rectangular Converstion Formulas

$$x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

10.10 Polar-Parametric Conversion Formulas

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

10.11 Slope of a Polar Curve

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta + r\sin\theta}$$

10.12 Area of a Polar Curve

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \; d\theta$$

10.13 Area Between Polar Curves

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_1^2 - r_2^2) \; d\theta$$

10.14 Length of a Polar Curve

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

10.15 Surface Area of a Polar Curve

Revolved about the x-axis: $SA = \int_{\alpha}^{\beta} 2\pi r \cdot \sin\theta \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta$

Revolved about the y-axis: $SA = \int_{\alpha}^{\beta} 2\pi r \cdot \cos\theta \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \; d\theta$

11 Appendix A: Trigonometric Identities

11.1 Reciprocal Identities

$$\sin u = \tfrac{1}{\csc u} \qquad \cos u = \tfrac{1}{\sec u}$$

$$\tan u = \tfrac{1}{\cot u} \qquad \sec u = \tfrac{1}{\cos u}$$

$$\csc u = \tfrac{1}{\sin u} \qquad \cot u = \tfrac{1}{\tan u}$$

11.2 Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u$$

$$1 + \cot^2 u = \csc^2 u$$

11.3 Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u$$

11.4 Even-Odd Identities

$$\sin\,(-u) = -\sin u$$

$$\cos\,(-u) = \cos u$$

$$\tan\,(-u) = -\tan u$$

11.5 Sum and Difference Formulas

$$\sin\,(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos\,(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan\,(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

11.6 Double Angle Formulas

$$\sin\,(2u) = 2\sin u\cos u$$
$$\cos\,(2u) = \cos^2 u - \sin^2 u$$
$$\cos\,(2u) = 2\cos^2 u - 1$$
$$\cos\,(2u) = 1 - 2\sin^2 u$$
$$\tan\,(2u) = \frac{2\tan u}{1-\tan^2 u}$$

11.7 Half Angle Formulas

$$\cos^2 u = \frac{1+\cos 2u}{2}$$
$$\sin^2 u = \frac{1-\cos 2u}{2}$$

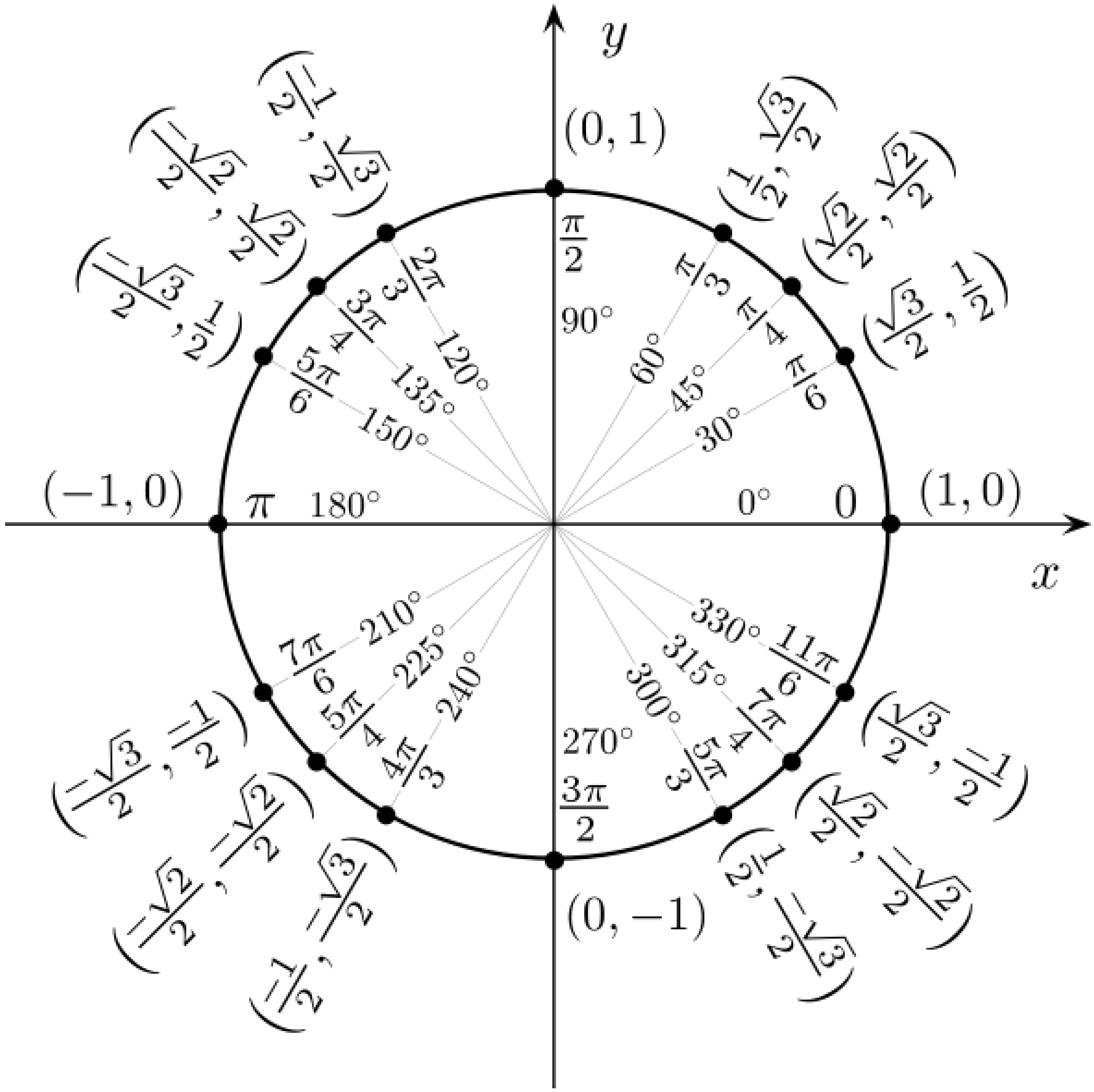
11.8 Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

11.9 Product-to-Sum Formulas

$$\sin u\sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u\cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u\cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u\sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

12 Appendix B: The Unit Circle



13 Appendix C: Common Indefinite Integrals

$\int x^n \; dx = \frac{x^n+1}{n+1} + C$

$\int x^{-1} \; du = \ln|x| + C$

$\int \sin ax \; dx = -\frac{1}{a}\cos ax + C$

$\int \cos ax \; dx = \frac{1}{a}\sin ax + C$

$\int \tan ax \; dx = \frac{1}{a}\ln|\sec ax| + C$

$\int \sec ax \; dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

$\int \csc ax \; dx = -\frac{1}{a}\ln|\csc ax - \cot ax| + C$

$\int \cot ax \; dx = \frac{1}{a}\ln|\sin ax| + C$

$\int \sec^2 x \; dx = \tan x + C$

$\int \csc^2 x \; dx = -\cot x + C$

$\int \sec x \tan x \; dx = \sec x + C$

$\int \csc x \cot x \; dx = -\csc x + C$

$\int e^{ax} \; dx = \frac{1}{a}e^{ax} + C$

$\int a^x \; dx = \frac{a^x}{\ln a} + C$

$\int \ln x \; dx = x\ln x - x + C$

$\int \log_{ax} \; dx = \frac{x\ln x - x}{\ln a} + C$

14 Appendix D: Don't Forget

USE THE CHAIN RULE